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SOME THEOREMS CONCERNING SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EXPRESSIONS

BY W. J. BERRY

THE investigations which led to the preparation of this paper, undertaken somewhat over a year ago at the suggestion of Professor Maxime Bôcher, were carried on, in part, under his direction, and the writer wishes to express his appreciation of the many illuminating criticisms received from him during the course of the work. Thanks are also due to Dr. Frank Irwin for permission to make use of his paper on Invariants of Linear Differential Expressions*, and for his kindness in reading and commenting upon the manuscript. Professor Elijah Swift has put the writer under obligations by helpful suggestions as to the form of presentation. The results obtained are in the nature of generalizations of theorems derived by Dr. Irwin in the paper referred to, and though the discussion is here limited to systems of linear partial differential expressions of the *second* order, the extension to systems of expressions of any order is immediate. The properties considered are five in number:—the existence of a system of multipliers; the existence of an adjoint, including the condition that a given system be self-adjoint; the existence of a Lagrange Identity; the existence of a Green's Theorem; and the existence of invariants.

The general system, L, of n second order linear partial differential expressions in n dependent and p independent variables may be written

$$L_i(u_1, \cdots, u_n) \equiv$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{r=1}^{p} l_{k,r}^{(i,j)} \frac{\partial^2 u_j}{\partial x_k\, \partial x_r} + \sum_{j=1}^{n}\sum_{k=1}^{p} l_k^{(i,j)} \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{n} l^{(i,j)} u_j, \quad i = 1, \cdots, n.$$

That the functions occurring in the course of the discussion may have all the necessary continuity, we shall demand:

1–) that in some p– dimensional region Ω of the space R_p defined by the independent variables, all the coefficients with double subscripts shall be continuous, with continuous partial derivatives of the first two orders with respect to each of their arguments, and with continuous second order cross derivatives of all possible types;

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* *Proc. Am. Acad. of Arts and Sciences*, vol. 44, no. 1, Nov. 1908.

2–) that all the coefficients with a single subscript shall be continuous in Ω, and shall there have continuous partial derivatives of the first order with respect to each of their arguments;

3–) that all coefficients without subscripts shall be continuous in Ω.

We shall further agree that hereafter throughout this paper, $l^{(i,j)}_{k,r}$ shall equal $l^{(i,j)}_{r,k}$ identically.

Definition. A set, v, of functions $v_i(x_1, \cdots, x_p)$, $i = 1, \cdots, n$, is said to form a system of multipliers for the system L, if

$$\sum_{i=1}^{n} v_i L_i \equiv \sum_{k=1}^{p} \frac{\partial S_k}{\partial x_k},$$

where

$$S_k \equiv \sum_{j=1}^{n} \sum_{r=1}^{p} s_r^{(k,j)} \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{n} s u_j^{(k,j)}$$

Concerning these coefficients we shall demand that they be functions of $x_1, \cdots, x_p$ continuous in Ω, and having there continuous partial derivatives of the first order with respect to each of their arguments.

The system L may also be written

$$L_i\ (u_1, \cdots, u_n) \equiv \sum_{j=1}^{n} L_{i,j}\ (u_j), \qquad i = 1, \cdots, n,$$

where

$$L_{i,j} \equiv \sum_{k=1}^{p} \sum_{r=1}^{p} l_{k,r}^{(i,j)} \frac{\partial^2 u_j}{\partial x_k \partial x_r} + \sum_{k=1}^{p} l_k^{(i,j)} \frac{\partial u_j}{\partial x_k} + l^{(i,j)} u_j.$$

The $L_{i,j}$ are, then, linear partial differential expressions of the second order in *one* dependent and p independent variables,—that is to say, expressions of precisely the sort treated by Dr. Irwin in his paper. If the system v is to form a set of multipliers for the system L, it is necessary that

$$\sum_{i=1}^{n} v_i L_i \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} v_i L_{i,j} \equiv \sum_{k=1}^{p} \frac{\partial S_k}{\partial x_k}.$$

Since this is an identity, the terms in u_j on the left must be identically equal to those on the right, hence

$$\sum_{i=1}^{n} v_i\, L_{i,j}(u_j) \equiv \sum_{k=1}^{p} \frac{\partial S_{k,j}}{\partial x_k},$$

where

$$S_{k,j} \equiv \sum_{r=1}^{p} s_r^{(k,j)} \frac{\partial u_j}{\partial x_r} + s u_j^{(k,j)}.$$

Equating coefficients from the two sides of this identity leads to the following equivalent set of necessary conditions for the existence of the system v:

$$2 \sum_{i=1}^{n} v_i\, l_{k,r}^{(i,j)} \equiv s_r^{(r,j)} + s_k^{(r,j)},$$

$$\sum_{i=1}^{n} v_i\, l_k^{(i,j)} \equiv \sum_{r=1}^{p} \frac{\partial s_k^{(r,j)}}{\partial x_r} + s^{(k,j)},$$

$$\sum_{i=1}^{n} v_i\, l^{(i,j)} \equiv \sum_{k=1}^{p} \frac{\partial s^{(k,j)}}{\partial x_k}, \qquad j = 1, \cdots n.$$

Operating on each identity of the first type with $\frac{\partial^2}{\partial x_k \partial x_r}$; on each identity of the second type with $-\frac{\partial}{\partial x_k}$; on each identity of the third type with 1, and adding all the resulting expressions for which j has a common value, gives the system of identities

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2}{\partial x_k \partial x_r} \left[l_{k,r}^{(i,j)} v_i \right] - \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\partial}{\partial x_k} \left[l_k^{(i,j)}\, v_i \right] + \sum_{i=1}^{n} l^{\,i,j)} v_i$$

$$\equiv \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2 s_r^{(k,j)}}{\partial x_k\, \partial x_r} - \sum_{k=1}^{p} \left\{ \sum_{r=1}^{p} \frac{\partial^2 s_r^{(k,j)}}{\partial x_k\, \partial x_r} + \frac{\partial s^{(k,j)}}{\partial x_k} \right\} + \sum_{k=1}^{p} \frac{\partial s^{(k,j)}}{\partial x_k} \equiv 0,$$

$$j = 1, \cdots, n.$$

The expressions at the left form a system, M, of second order linear partial differential expressions

$$M_i(v_1, \cdots, v_n)$$

$$\equiv \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2}{\partial x_k \partial x_r} \left[l_{k,r}^{(j,i)} v_j \right] - \sum_{j=1}^{n} \sum_{k=1}^{p} \frac{\partial}{\partial x_k} \left[l_k^{(j,i)} v_j \right] + \sum_{j=1}^{n} l^{(j,i)} v_j$$

$$\equiv \sum_{j=1}^{n} M_{i,j}(v_j),$$

$$i = 1, \cdots, n,$$

where $M_{i,\,j}\,(v_j)$ is the adjoint of $L_{j,\,i}\,(u_i)$ under the definition laid down by Dr. Irwin.*

Definition. The system of second order linear partial differential expressions M is said to be the *adjoint system* of the system L.

It is necessary, then, in order that the system v form a system of multipliers for the system L, that it be a set of solutions of the system of differential equations

$$M_i\,(v_1, \cdots, v_n) = 0, \qquad i = 1, \cdots, n.$$

This necessary condition is also sufficient, for if we have given the v_i as solutions of this system of differential equations, and choose certain of the coefficients $s_r^{(k,j)}$ at will, say those for which $k > r$†, then it will be possible to determine all the other coefficients so as to satisfy all the necessary conditions of the first two types.

It follows that

$$\sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2}{\partial x_k\, \partial x_r} \left[l_{k,\,r}^{(j,\,i)} v_j \right] - \sum_{j=1}^{n} \sum_{k=1}^{p} \frac{\partial}{\partial x_k} \left[l^{(j,i)} v_j \right] \equiv - \sum_{k=1}^{p} \frac{\partial s^{(k,i)}}{\partial x_k}, \qquad i = 1, \cdots, n,$$

and the necessary conditions of the third type are automatically satisfied. We are thus led to

Theorem 1. A necessary and sufficient condition that the system of functions v form a set of multipliers for the system of second order linear partial differential expressions L, is that it be a set of solutions of the system of second order linear partial differential *equations*

$$M_i = 0, \qquad i = 1, \cdots, n,$$

in which the left-hand members are the expressions of the system M, adjoint to L.

The idea of "multiplier", as here defined, represents the extension to systems of second order linear *partial* differential *expressions* of analogous concepts introduced by Jacobi in connection with systems of first order linear homogeneous *ordinary* differential *equations*‡.

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* Loc. cit., page 8.

† This is, of course, not the only choice possible, but it agrees with that made by Dr. Irwin in the simpler case; loc. cit., page 9.

‡ *Encyklopädie der Math. Wiss.*, Bd. II, A 4 b, § 12.

The adjoint system, consisting, like the original system, of second order linear partial differential expressions, may be written with its own coefficients,

$$M_i(v_1,\cdots,v_n)\equiv\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{r=1}^{p}m_{k,r}^{(i,j)}\frac{\partial^2 v_j}{\partial x_k\,\partial x_r}+\sum_{j=1}^{n}\sum_{k=1}^{p}m_k^{(i,j)}\frac{\partial v_j}{\partial x_k}+\sum_{j=1}^{n}m^{(i,j)}v_j$$

$$i=1,\cdots,n,$$

these coefficients being connected with those of L by the relations

$$m_{k,r}^{(i,j)}\equiv l_{k,r}^{(j,i)},$$

$$m_k^{(i,j)}\equiv 2\sum_{r=1}^{p}\frac{\partial l_{k,r}^{(j,i)}}{\partial x_r}-l_k^{(j,i)},$$

$$m^{(i,j)}\equiv\sum_{k=1}^{p}\sum_{r=1}^{p}\frac{\partial^2 l_{k,r}^{(j,i)}}{\partial x_k\partial x_r}-\sum_{k=1}^{p}\frac{\partial l_k^{(j,i)}}{\partial x_k}+l^{(j,i)},$$

or in more symmetrical form,

$$m_{k,r}^{(i,j)}\equiv l_{k,r}^{(j,i)}\,,$$

$$\sum_{r=1}^{p}\frac{\partial m_{k,r}^{(i,j)}}{\partial x_r}-m_k^{(i,j)}\equiv-\sum_{r=1}\frac{\partial l_{k,r}^{(j,i)}}{\partial x_r}+l_k^{(j,i)},$$

$$\sum_{k=1}^{p}\frac{\partial m_k^{(i,j)}}{\partial x_k}-2m^{(i,j)}\equiv\sum_{k=1}^{p}\frac{\partial l_k^{(j,i)}}{\partial x_k}-2l^{(i,i)}\,.$$

This same symmetry shows that the adjoint relation is mutual,—that is to say, if the system M is adjoint to L, L is also adjoint to M.

Definition. By a self-adjoint system we shall agree to mean one such that when its adjoint has been formed the coefficients will be found to be identical with those of the original system.

The truth of the following theorem is at once evident.

Theorem 2. Necessary and sufficient conditions that a system L of second order linear partial differential expressions be self-adjoint are:

1) $l^{(i,j)}\equiv l^{(j,i)}$, whatever the lower indices;

2) $l_k^{(i,j)}\equiv\displaystyle\sum_{r=1}^{p}\frac{\partial l_{k,r}^{(i,j)}}{\partial x_r}$, $\qquad i,j=1,\cdots,n;\ k=1,\cdots,p.$

For the special case in which the coefficients of L are constants, this second condition becomes $l_k^{(i,j)}\equiv 0$,

$$\sum_{i=1}^{n}\left\{v_iL_i(u_1,\cdots,u_n)-u_iM_i(v_1,\cdots,v_n)\right\}\equiv$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{v_{j}L_{ji}(u_i)-u_iM_{i,j}(v_j)\right\}.$$

Since $L_{j,i}(u_i)$ and $M_{i,j}(v_j)$ are adjoint expressions they satisfy the relation

$$v_jL_{j,i}(u_i)-u_iM_{i,j}(v_j)\equiv\sum_{k=1}^{p}\frac{\partial R_k^{(i,j)}}{\partial x_k},\qquad i,j=1,\cdots,n,$$

where the $R_k^{(i,j)}$ are bilinear in u_i and v_j and their partial derivatives of the first order.* Hence

$$\sum_{i=1}^{n}\left\{v_iL_i-u_iM_i\right\}\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{p}\frac{\partial R_k^{(i,j)}}{\partial x_k}\equiv\sum_{k=1}^{p}\frac{\partial R_k}{\partial x_k},$$

where the R_k are bilinear in all the u_i and v_i and their partial derivatives of the first order.

Definition. A relation of the form

$$\sum_{i=1}^{n}\left\{v_iL_i-u_iM_i\right\}\equiv\sum_{k=1}^{p}\frac{\partial R_k}{\partial x_k},$$

in which L and M are adjoint systems and the functions R_k are bilinear in the u_i and v_i and their partial derivatives of the first order, is said to be *Lagrange's Identity* for the systems u and v.

It will be noted that Lagrange's Identity furnishes a new and simple proof for the second part of Theorem 1, for if the system v is a set of solutions of $M_i(v_1,\cdots,v_n)=0,\quad i=1,\cdots,n,$ then

$$M_i(v_1,\cdots,v_n)\equiv 0,\quad\text{and}\quad\sum_{i=1}^{n}v_iL_i\equiv\sum_{k=1}^{p}\frac{\partial R_k}{\partial x_k},$$

where the R_k satisfy all the conditions imposed on the S_k of the earlierr poofs. Therefore v is a system of multipliers for L, and, from the symmetry of Lagrange's Identity, u is a system of multipliers for M.

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* See Irwin loc. cit., page 10.

Theorem 3. If between two systems, L and N, each consisting of n second order linear partial differential expressions in n dependent and p independent variables, there exists a relation of the form of Lagrange's Identity, L and N are adjoint systems.

For, let M be the system adjoint to L, then

$$\sum_{i=1}^{n} \left\{ v_i L_i(u_1, \cdots, u_n) - u_i M_i(v_1, \cdots, v_n) \right\} \equiv \sum_{k=1}^{p} \frac{\partial R_k}{\partial x_k}.$$

By hypothesis

$$\sum_{i=1}^{n} \left\{ v_i L_i(u_1, \cdots, u_n) - u_i N_i(v_1, \cdots, v_n) \right\} \equiv \sum_{k=1}^{p} \frac{\partial T_k}{\partial x_k},$$

$$\sum_{i=1}^{n} u_i(N_i - M_i) \equiv \sum_{k=1}^{p} \left(\frac{\partial R_k}{\partial x_k} - \frac{\partial T_k}{\partial x_k}\right) \equiv \sum_{k=1}^{p} \frac{\partial}{\partial x_k} (R_k - T_k).$$

The system u is accordingly a system of multipliers for $(N - M)$. If $\mathfrak{M}$ be the adjoint system to $(N - M)$, u is a set of solutions for $\mathfrak{M}$. But the u_i are entirely arbitrary; hence $\mathfrak{M}_i \equiv 0$, and consequently

$$N_i - M_i \equiv 0, \qquad i = 1, \cdots, n,$$

which proves the theorem.

Definition. The expression obtained by integrating Lagrange's Identity,

$$\int\int_{\Omega}\cdots\int \left\{ \sum_{i=1}^{n} (v_i L_i - u_i M_i) \right\} dx_1 \cdots dx_p \equiv \int_S \sum_{k=1}^{p} R_k \cos a_k \, dS,$$

is defined as a Green's Theorem for the system L, S being the $(p - 1)$-way spread which is the boundary of the region Ω.

If the system L is self-adjoint, the expression $L_{i,j}(u_j)$ is self-adjoint. Such an expression satisfies a three term form of Lagrange's Identity:*

$$v_j L_{j,i}(u_i) - \sum_{k=1}^{p} \frac{\partial P_k^{(i,j)}}{\partial x_k} \equiv u_i M_{i,j}(v_j) - \sum_{k=1}^{p} \frac{\partial Q_k^{(i,j)}}{\partial x_k}$$

$$\equiv - \sum_{k=1}^{p} \sum_{r=1}^{p} l_{k,r}^{(i,j)} \frac{\partial v_j}{\partial x_k} \frac{\partial u_i}{\partial x_r} + l^{(i,j)} v_j u_i,$$

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* See Irwin, loc. cit., page 11.

where

$$P_k^{(i,j)} \equiv \sum_{r=1}^{p} l_{k,r}^{(i,j)} v_j \frac{\partial u_i}{\partial x_r}; \quad Q_k^{(i,j)} \equiv \sum_{r=1}^{p} l_{k,r}^{(i,j)} u_j \frac{\partial v_i}{\partial x_r}, \quad i, j = 1, \cdots, n.$$

Thus we have

Theorem 4. If L be a self-adjoint system of second order linear partial differential expressions, there is a three term form of Lagrange's Identity:

$$\sum_{i=1}^{n} v_i L_i - \sum_{k=1}^{p} \frac{\partial P_k}{\partial x_k} \equiv \sum_{i=1}^{n} u_i M_i - \sum_{k=1}^{p} \frac{\partial Q_k}{\partial x_k}$$

$$\equiv -\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{p}\sum_{k=1}^{p} l_{k,r}^{(i,j)} \frac{\partial v_j}{\partial x_k}\frac{\partial u_i}{\partial x_r} + \sum_{i=1}^{n}\sum_{j=1}^{n} l^{(i,j)} u_i v_j,$$

where

$$P_k \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} P_k^{(i,j)} \equiv \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{p} l_{k,r}^{(i,j)} v_j \frac{\partial u_i}{\partial x_r},$$

$$Q_k \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} Q_k^{(i,j)} \equiv \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{p} l_{k,r}^{(i,j)} u_j \frac{\partial v_i}{\partial x_r},$$

the integration of which leads to a corresponding three term form of Green's Theorem.

Consider the transformation, T, defined by

$$u_j = \sum_{s=1}^{n} \psi_{j,s}\eta_s, \qquad |\psi_{1,1} \cdots \psi_{n,n}| \neq 0, \qquad j = 1, \cdots, n,$$

where the $\psi_{j,s}$ are functions of $x_1, \cdots, x_p$, which, together with all their partial derivatives of the first two orders and all their cross derivatives of the second order, are continuous in the region Ω. The functions η_s are to satisfy all the conditions imposed on the u_j. By this transformation the system L is carried over into the system Λ:

$$\Lambda_i(\eta_1, \cdots, \eta_n) \equiv \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{r=1}^{p} \lambda_{k,r}^{(i,j)} \frac{\partial^2 \eta_j}{\partial x_k \partial x_r} + \sum_{j=1}^{n}\sum_{k=1}^{p} \lambda_k^{(i,j)} \frac{\partial \eta_j}{\partial x_k} + \sum_{j=1}^{n} \lambda^{(i,j)} \eta_j,$$

$$i = 1, \cdots, n.$$

The formulae for the new coefficients are*

$$\lambda^{(i,j)}_{k,r} \equiv \sum_{s=1}^{n} l^{(i,s)}_{k,r} \psi_{s,j},$$

$$\lambda^{(i,j)}_{k} \equiv 2\sum_{s=1}^{n} \sum_{r=1}^{p} l^{(i,s)}_{k,r} \frac{\partial \psi_{s,j}}{\partial x_r} + \sum_{s=1}^{n} l^{(i,s)}_{k} \psi_{s,j},$$

$$\lambda^{(i,j)} \equiv \sum_{s=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} l^{(i,s)}_{k,r} \frac{\partial^2 \psi_{s,j}}{\partial x_k \partial x_r} + \sum_{s=1}^{n} \sum_{k=1}^{p} l^{(i,s)}_{k} \frac{\partial \psi_{s,j}}{\partial x_k} + \sum_{s=1}^{n} l^{(i,s)} \psi_{s,j},$$

$$i = 1, \cdots, n.$$

Definition. A set of functions, I, of the coefficients of a given system of differential expressions, L, and their derivatives, shall be called a *relatively invariant set of degree* μ, with respect to the transformation T, if every one of a similar set of functions, built up from the coefficients of the transform Λ and their derivatives, is expressible, by means of the above relations, as a linear combination of the functions of the original set, with coefficients of the form $\prod_{i,j} \psi_{i,j}^{a_{i,j}}$, where $\sum_{i,j} a_{i,j} = \mu$. If the coefficients in the combination are all unity, I shall be called an *absolutely invariant set.*

It will be found convenient to refer to the functions in I as the *elements* of the set.

If the transformation T be applied to Lagrange's Identity, the latter goes over into

$$\sum_{i=1}^{n} \left\{ v_i \Lambda_i - \eta_i \sum_{s=1}^{n} \psi_{s,i} M_s \right\} \equiv \sum_{k=1}^{p} \frac{\partial \overline{R}_k}{\partial x_k},$$

where the $\overline{R}_k$ are bilinear in the v_i and η_i and their partial derivatives of the first order. Hence Λ_i and

$$\sum_{s=1}^{n} \psi_{s,i} M_s, \qquad i = 1, \cdots, n,$$

are adjoint systems.

Theorem 5. If the dependent variables u_j, $j = 1 \cdots, n$, in the system of second order linear partial differential equations, L, be subjected to the transformation T, defined by the equations

$$u_j = \sum_{s=1}^{n} \psi_{j,s}(x_1, \cdots, x_p)\eta_s, \qquad j = 1, \cdots, n,$$

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* Compare Irwin, loc. cit., page 18.

the expressions in $\mathfrak{M}$, the system adjoint to Λ, the transform of L under the transformation T, are linear combinations of the expressions in M, the system adjoint to L, the coefficients of the combinations being the $\psi_{j,s}$; — that is

$$\mathfrak{M}_i \equiv \sum_{s=1}^{n} \psi_{s,i} M_s, \qquad i = 1, \cdots, n.$$

It is at once evident that the coefficients of the system M, adjoint to the given system L, are the elements of a relatively invariant set of degree one under the transformation T.

Definition. Two configurations are said to be equivalent with respect to a given set of transformations, if there is one transformation of the set which carries the first configuration over into the second, and another which carries the second into the first.

Definition. Given two configurations, C and C', whose corresponding relatively invariant sets under a set of transformations T are I and I', then, if there be a set of functions Θ, of the independent variables, such that when the elements of I' are linear combinations of the elements of I with coefficients taken from Θ, the configurations C and C' are equivalent, the invariant set I is said to be *complete*.

In the case of absolute invariants all the elements of Θ are unity.

Let the configurations be the systems of second order linear partial differential expressions

$$L_i\, u(_1, \cdots, u_n)\,; \qquad \mathrm{A}_i\, (\eta_1, \cdots, \eta_n), \quad i = 1, \cdots, n,$$

whose adjoint systems are, respectively,

$$M_i\, (v_1, \cdots, v_n)\,; \quad \mathfrak{M}_i\, (v_1, \cdots, v_n), \qquad i = 1, \cdots, n.$$

Let Θ be composed of the functions

$$\theta_{i,j}\, (x_1, \cdots, x_p), \quad i,j = 1, \cdots, n.$$

Now a known invariant set of degree one for either configuration under a transformation T of the form

$$u_i = \sum_{j=1}^{n} \psi_{i,j}\, \eta_j\,; \qquad | \psi_{1,1} \cdots, \quad \psi_{n,n} | \not\equiv 0 \;; \quad i = 1, \cdots, n,$$

has for its elements the coefficients of the adjoint system. Suppose the coefficients of $\mathfrak{M}$ to be linear combinations of the corresponding coefficients of M, the coefficients of the combination being taken from Θ so that

$$\mathfrak{M}_i \equiv \sum_{j=1}^{n} \theta_{i,j} M_j, \qquad i = 1, \cdots, n.$$

Subject L to the transformation

$$u_i = \sum_{j=1}^{n} \theta_{i,j} \eta_j, \qquad i = 1, \cdots, n,$$

and assume the $\theta_{i,j}$ to have been so chosen as to satisfy the relation $|\theta_{1,1}, \cdots, \theta_{n,n}| \not\equiv 0$. The transformation is then of the form T, and L is carried over into $\overline{\Lambda}$, whose adjoint is

$$\sum_{j=1}^{n} \theta_{i,j} M_j \equiv \mathfrak{M}_i, \quad i = 1, \cdots, n.$$

But $\overline{\Lambda}$ and Λ, having the same adjoint, are identical, hence there is one transformation of the form T which carries L into Λ. The inverse transformation exists and is also of the form T. In a similar fashion it may be proved to carry Λ into L. Hence:

Theorem 6. The coefficients of a system of second order linear partial differential expressions M, adjoint to a given system L, are the elements of a relatively invariant set of degree one under any transformation of the form T, and this relatively invariant set is complete.

Definition. By the partial weight of any coefficient of the system L with respect to any one of the independent variables, shall be meant the number of lower indices which refer to that variable. The $\psi_{i,j}$ shall all be of weight zero with respect to each, that is every one, of the independent variables. To find the partial weight of a partial derivative of one of the above functions with respect to any one of the independent variables, diminish the partial weight of the function itself with respect to the variable in question by the number of times it has been differentiated with respect to that variable. The total weight of any of the above functions shall be the sum of its partial weights. The weight (total or partial) of a product shall be equal to the sum of the weights (total or partial) of its factors. A polynomial is isobaric, totally or partially, if all of its terms are of the same total or partial weight.

With this convention as to weights, an obvious extension of the methods employed by Dr. Irwin* to establish his propositions 5 and 12 leads to the following theorems.

Theorem 7. An element of an invariant set may or may not be isobaric; but if it is not, it is merely the sum of elements which are.

This theorem applies to the total weight or to any one of the partial weights.

Theorem 8. An element of an invariant set of functions built up from the coefficients of a system of second order linear partial differential expressions, adjoint to a given system L, if not itself homogeneous in each and every coefficient present and its derivatives, is merely the sum of elements which are.

Theorem 9. An element of an invariant set of functions built up from the coefficients of the adjoint system, if it is homogeneous of degrees ν in some one coefficient and its derivatives, and contains no other coefficient, is essentially nothing but the νth power of the coefficient present.

By means of theorems 7, 8 and 9 may be proved our final theorem.

Theorem 10. Essentially the only relatively invariant set of degree one for the system L under the transformation T, has for its elements the coefficients of the system adjoint to L.

Under the special transformation $u_i = \psi_i \eta_i$, $i = 1, \cdots, n$, the invariant set becomes a set of invariant functions* to which Irwin's theorems apply almost verbatim.

In addition to the article in the *Encyklopädie*, to which reference has already been made, special cases of certain theorems in this paper are developed by Professor Max Mason in his article on "Green's Theorems and Green's Functions for Certain Systems of Differential Equations",† and by Jordan in his *Cours d'Analyse* (ed. 1887, tome 3, p. 142, § 115).

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* Loc. cit., p. 19, and pp. 25, 26.

*See Irwin, loc. cit., p. 18.

† *Trans. Am. Math. Soc.*, vol. 5 (1904), p. 220.